Simple relativistic model of a finite-size particle

Iwo Bialynicki-Birula

*Centrum Fizyki Teoretycznej PAN Lotników 32/46, 02-668 Warsaw, Poland**
and Institut für Theoretische Physik, Johann Wolfgang Goethe-Universität
Robert-Mayer-Strasse 8-10, Frankfurt am Main, Germany

Abstract

Soluble model of a relativistic particle describing a bag of matter with fixed radius held together in perfect balance by a self-consistent combination of three forces generated by electromagnetic and massive scalar and vector fields is presented. For realistic values of parameters the bag radius becomes that of a proton.

* Permanent address. E-mail address: birula@planif61.bitnet

1. Introduction

"I never satisfy myself until I can make a mechanical model of a thing". These words of Lord Kelvin [1] describe best my motivation to search for a simple relativistic mechanical model of an extended charged particle. Since over the years the concept of a mechanical model has undergone a substantial evolution, I do not hesitate to include also fields as building blocks for "mechanical" models.

In this Letter I present a very simple model of a relativistic charged object (one may think of it as a proton or a nucleus) that is solvable in terms of elementary functions. The model consists of a swarm of particles endowed with three types of charges e, g_S, and g_V interacting in a self-consistent manner with three relativistic fields. The constituent particles are occupying a bounded region in space (a bag) and are described by a scalar phase-space distribution function $f(\mathbf{r}, \mathbf{p}, t)$. The solution of the field equations in this model can also be given a hydrodynamic interpretation in which case the bag becomes a droplet of pressureless fluid interacting with the three fields. An unexpected result of this investigation is the appearance of a quantization condition from which the radius of the bag (or the droplet) is determined.

Almost a century ago Poincaré [2] argued that to counterbalance electrostatic repulsion inside a charged particle one must introduce cohesive forces — Poincaré stresses. In order to comply with the requirements of relativity theory, these stresses must possess a dynamics of their own, they must in Pauli's words "depend on physical quantities which are causally determined by differential equations" [3]. Nowadays we know that a possible candidate for a "causally determined physical quantity" that will hold a charged particle together is a scalar field. For unlike the electrostatic forces generated by vector fields, the forces between like charges generated by scalar fields are attractive. To prevent collapse one needs also a short range repulsion and that in turn can be supplied by a massive vector field. That explains the choice of the main ingredients of my model. They are, of course, being used quite often in particle physics and in nuclear physics [4].

Several classical models of a charged particle have been invented in the past (see, for example, [5–7] and also the reviews of the classical electron theory [8–11]), but in all of them the Poincaré stresses were introduced ad hoc. The relativistic model of a classical charged particle with a finite, sharply defined radius presented here is the first, to my knowledge, in which the Poincaré stresses incorporate some realistic elements. It can be used in two ways. First, with its help one may illustrate and clarify some old problems of relativistic theories of extended objects, discussed already by Abraham [12], Poincaré [2], Lorentz [13], and von Laue [14,15], that also more recently continued to cause controversy [16,17]. Second, with the proper choice of parameters, the model may serve as a zeroth order approximation in theories of nuclei and their high-energy collisions. For it is clear today that if we are to use in realistic applications a semi-classical model of a relativistic extended object it will not be to describe an electron as Poincaré and others had tried in vain. We may still try, however, to describe in this manner nucleons and nuclei whose quantum features are less predominant since their Compton wave lengths are much smaller than their physical dimensions.

2. Description of the model

The starting point of my construction is the following set of relativistic equations describing the motion of a relativistic dust of particles interacting in a self-consistent manner with the three fields ($c = 1$)

$$[(m - g_S\phi)(\partial_t + \mathbf{v}\cdot\nabla) + m\mathbf{F}\cdot\boldsymbol{\partial_p}]\, f(\mathbf{r},\mathbf{p},t) = 0, \tag{1}$$

$$\partial_\mu F^{\mu\nu} = ej^\nu, \tag{2}$$

$$(\Box + m_S^2)\phi = g_S\rho, \tag{3}$$

$$\partial_\mu G^{\mu\nu} + m_V^2 W^\nu = g_V j^\nu. \tag{4}$$

where ∇ and $\boldsymbol{\partial_p}$ denote the derivatives with respect to $\mathbf{r}$ and $\mathbf{p}$, respectively. The laboratory-frame three-velocity $\mathbf{v}$ and the three-force $\mathbf{F}$ are related to the spatial parts of the four-velocity u^μ and the four-force f^μ,

$$\begin{aligned} u^\mu &= p^\mu/m, \\ f^\mu &= eF^{\mu\nu}u_\nu - g_S(\partial^\mu - u^\mu u^\nu\partial_\nu)\phi + g_V G^{\mu\nu}u_\nu, \end{aligned} \tag{5}$$

in the usual way, namely, $v^i = u^i/u^0 = p^i/E_p$, $F^i = f^i/u^0$. The antisymmetric field tensors $F_{\mu\nu}$ and $G_{\mu\nu}$ are constructed in the standard manner from the corresponding potential four-vectors A_μ and W_μ. The four-current j^μ and the scalar density ρ are defined in terms of the distribution function as follows

$$j^\mu(\mathbf{r},t) = \int \frac{d^3p}{E_p} p^\mu f(\mathbf{r},\mathbf{p},t), \tag{6}$$

$$\rho(\mathbf{r},t) = \int \frac{d^3p}{E_p} m f(\mathbf{r},\mathbf{p},t). \tag{7}$$

The set of coupled equations (1)–(4) is a generalization of Vlasov-Maxwell equations used in plasma physics. Despite the nonrelativistic appearance of Eq.(1), full relativistic covariance of the theory based on these equations can be proven in the same manner as we have done earlier [18] for the pure Vlasov-Maxwell theory.

Equations (1)–(4) may be viewed as classical if one assigns the parameters m_S and m_V a dimension of inverse length and uses them only as measures of the range of forces due to scalar and vector fields. Planck's constant will make its appearance at the end in Eqs.(39) and (40) when the parameters of the model are expressed in terms of the masses of mesons and their coupling constants.

3. A static solution of the model

A solution describing a bag of finite size is most easily obtained in the static case, when all constituent particles are at rest,

$$f(\mathbf{r},\mathbf{p},t) = \delta(\mathbf{p})\,\rho(\mathbf{r}), \tag{8}$$

and the field equations (2)–(4) reduce to the following simple set (in the static case there is just one scalar density ρ since $E_p = m$)

$$-\Delta A_0 = e\rho, \tag{9}$$
$$(-\Delta + m_S^2)\phi = g_S\rho, \tag{10}$$
$$(-\Delta + m_V^2)W_0 = g_V\rho. \tag{11}$$

In order to satisfy also the equation (1) for the distribution function (8), I assume the following equilibrium condition

$$\rho\mathbf{F} \equiv -\rho\nabla(eA_0 - g_S\phi + g_V W_0) = 0. \tag{12}$$

This condition simply means that the net force acting on each particle in the bag vanishes. Note that the equilibrium condition is not imposed everywhere in space — that would lead to a trivial solution — but only in those regions of space where matter is present. That is why the bag's boundary must be well defined. The equilibrium condition severely restricts possible solutions, but fortunately it does leave room for some interesting ones. Since the solutions of the equations (9)–(12) can always be scaled, I shall normalize them by imposing the following normalization condition

$$\int d^3r\,\rho = 1. \tag{13}$$

A simple, spherically symmetric solution of Eqs.(9)–(11) is written below separately for the inside and the outside of the bag.

Inside ($r \leq R$):

$$\rho = f_+ - f_-, \tag{14}$$
$$eA_0 = e^2\Big(\frac{f_+}{k_+^2} - \frac{f_-}{k_-^2}\Big) - V_0, \tag{15}$$
$$g_S\phi = g_S^2\Big(\frac{f_+}{k_+^2 + m_S^2} - \frac{f_-}{k_-^2 + m_S^2}\Big), \tag{16}$$
$$g_V W_0 = g_V^2\Big(\frac{f_+}{k_+^2 + m_V^2} - \frac{f_-}{k_-^2 + m_V^2}\Big), \tag{17}$$

Outside ($r > R$):

$$\rho = 0, \tag{18}$$
$$eA_0 = \frac{e^2}{4\pi r}, \tag{19}$$
$$g_S\phi = \frac{b_S}{4\pi r}e^{-m_S(r-R)}, \tag{20}$$
$$g_V W_0 = \frac{b_V}{4\pi r}e^{-m_V(r-R)}, \tag{21}$$

where b_S, b_V, and V_0 are constants and $f_\pm$ are the following S-wave solutions of the Helmholtz equations,

$$f_\pm = \frac{d_\pm}{4\pi} \frac{\sin(k_\pm r)}{r}. \tag{22}$$

The wave vectors $k_\pm$ are determined from a biquadratic equation obtained from the equilibrium condition (12),

$$k_\pm^2 = \frac{B \pm \sqrt{D}}{2}, \tag{23}$$

where

$$Q^2 = e^2 - g_S^2 + g_V^2, \tag{24}$$

$$D = B^2 - 4e^2 Q^2 m_S^2 m_V^2, \tag{25}$$

and

$$B = (g_S^2 - e^2) m_V^2 - (g_V^2 + e^2) m_S^2. \tag{26}$$

The remaining parameters b_S, b_V, $d_\pm$, the depth of the potential well V_0, and the radius of the bag R are determined from six continuity conditions at the bag's boundary. A combination of these conditions gives the following quantization condition for R, very similar to those arising in wave mechanics,

$$T(m_S) = T(m_V), \tag{27}$$

where

$$T(m_X) = \frac{k_+^2 + m_X^2}{k_-^2 + m_X^2} \frac{k_- + m_X \tan(k_- R)}{k_+ + m_X \tan(k_+ R)}. \tag{28}$$

Eq.(27) has infinitely many solutions for R, but only the lowest one is physically acceptable because all higher ones do not lead to a positive density ρ. Once the radius of the bag is determined, the remaining parameters can be calculated from explicit formulas. In particular,

$$V_0 = \frac{e^2}{4\pi} \frac{T k_+ - k_+^2/k_-}{T t_+ - k_+^2 t_-/k_-^2}, \tag{29}$$

where $t_\pm = \tan(k_\pm R) - k_\pm R$ and T is the common value of $T(m_S)$ and $T(m_V)$.

The solutions of field equations obtained in this way have several general features that are worth noting. The bag has a sharp edge — the value of the density at the surface of the bag is always finite,

$$\rho(R) = \frac{e^2 m_S m_V}{4\pi Q^2 R}. \tag{30}$$

It is seen from this formula that for a solution to exist the effective repulsion must be stronger than attraction ($Q^2 > 0$).

4. Hydrodynamic description

My bag model may easily be converted into a droplet model by replacing the description in terms of the distribution function by a hydrodynamic description in terms of a density scalar field ρ and a four-velocity field u^μ that characterize the state of the fluid. The starting point of the hydrodynamic description is the following set of relativistic equations

$$\partial_\mu(\rho u^\mu) = 0, \tag{31}$$

$$(m - g_S\phi)\rho u^\nu\partial_\nu u^\mu = e\rho F^{\mu\nu}u_\nu - g_S\rho(\partial^\mu\phi - u^\mu u^\nu\partial_\nu\phi) + g_V\rho G^{\mu\nu}u_\nu, \tag{32}$$

$$\partial_\mu F^{\mu\nu} = e\rho u^\nu, \tag{33}$$

$$(\Box + m_S^2)\phi = g_S\rho, \tag{34}$$

$$\partial_\mu G^{\mu\nu} + m_V^2 W^\nu = g_V\rho u^\nu. \tag{35}$$

These equations describe matter modelled by a pressureless fluid of density ρ moving with four-velocity u^μ. The fluid is endowed with the three types of charges e, g_S, and g_V and is interacting in a self-consistent manner with the three relativistic fields. For a fluid without pressure the static solution remains the same as in the bag model. One may also add the pressure term and an equation of state and seek numerical solutions of the same general nature — with a sharply defined radius.

The system described by the equations (31)–(35) is conservative (and that is also true for the system described by generalized Vlasov-Maxwell equations (1)–(4)) since the equations of motion guarantee that the total energy-momentum tensor $T^{\mu\nu}$ of the system,

$$\begin{aligned} T^{\mu\nu} &= (m - g_S\phi)\rho u^\mu u^\nu + F^{\mu\lambda}F_\lambda{}^\nu + \tfrac{1}{4}g^{\mu\nu}F_{\lambda\rho}F^{\lambda\rho} + \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\tfrac{1}{2}(\partial_\lambda\phi\partial^\lambda\phi - m_S^2\phi^2) \\ &\quad + G^{\mu\lambda}G_\lambda{}^\nu + m_V^2 W^\mu W^\nu + g^{\mu\nu}(\tfrac{1}{4}G_{\lambda\rho}G^{\lambda\rho} - \tfrac{1}{2}m_V^2 W_\lambda W^\lambda), \end{aligned} \tag{36}$$

is conserved, $\partial_\mu T^{\mu\nu} = 0$.

The droplet (or the bag) described here is energetically stable. The combined energy of the three fields, owing to the equilibrium condition, reduces to half of the interaction energy with the scalar field (a version of the virial theorem) and that leads to the following value of the total energy of the system obtained from (36) by integrating the energy density

$$\begin{aligned} E_{tot} &= \int d^3r(m - g_S\phi)\rho + \tfrac{1}{2}\int d^3r[(\nabla A_0)^2 + (\nabla\phi)^2 + m_S^2\phi^2 + (\nabla W_0)^2 + m_V^2 W_0^2] \\ &= \int d^3r(m - g_S\phi)\rho + \tfrac{1}{2}\int d^3r[-A_0\Delta A_0 - \phi\Delta\phi + m_S^2\phi^2 - W_0\Delta W_0 + m_V^2 W_0^2] \\ &= m + \tfrac{1}{2}\int d^3r[eA_0\rho - g_S\phi\rho + g_V W_0\rho] = m - V_0/2. \end{aligned} \tag{37}$$

This formula conveys an important message concerning equilibrium configurations in local field theories. In order to calculate the total energy of the matter-field configuration in the whole space it is enough to know the field inside the bag. Since we know the relativistic transformation properties of the fields describing the model, a static solution may easily be boosted to give a model of a moving particle. The infamous factor of 4/3 discovered by J.J.Thompson [19] that was plaguing all naive electron models does not appear here. It can be checked by an explicit calculation or inferred from a generalized virial theorem (cf., for example, [7]) that my solution satisfies the von Laue's condition [15]

$$\int d^3r\, T^{ii}(\mathbf{r}) = 0, \tag{38}$$

that guarantees proper transformation properties of the energy-momentum vector of the particle.

5. A tentative connection with reality

Now I come to the problem of assigning definite values to the parameters of the model in order to see if the bag can be made to resemble a nucleon. I have not done any elaborate parameter fitting, since that will definitely require the introduction of several scalar and vector mesons as is done in modern mean-field theories of nuclear structure [4]. I have just taken as my input the same values of the masses of the σ-meson and the ω-meson and the values of the coupling constants that are used in the simplest version (QHD-I) of the mean field theory of nuclear matter (Ref. [4],p. 125),

$$m_S = 550 \text{ MeV}/\hbar c, \quad m_V = 783 \text{ MeV}/\hbar c, \tag{39}$$

$$g_S^2 = 91.64\, \hbar c, \qquad g_V^2 = 136.2\, \hbar c. \tag{40}$$

The quantization condition (27) gives then the following values for the radius of the bag and the depth of the binding potential

$$R = 1.05 \text{ fm}, \quad V_0 = 31.42 \text{ MeV}. \tag{41}$$

The values of the proton mean radius and the binding energy calculated for these values are

$$r_p = 0.714 \text{ fm}, \quad E_B = 15.71 \text{ MeV}, \tag{42}$$

and they are to be compared with the experimentally measured radius 0.862 fm and the bulk binding energy per nucleon in nuclear matter 15.75 MeV. I am very far from advocating the use of this model in its present rudimentary form as a realistic model of nuclear structure and I am presenting these numbers only to show that they are not completely out of touch with reality.

To illustrate my results I show in Fig. 1 separately the curves for the three potentials, in Fig. 2 the combined effective potential well, and in Fig. 3 the distribution of matter in the bag. All these graphs correspond to the choice of parameters given by the formulas (39) and (40).

The potential well shown Fig. 2 may be used as an initial step in constructing a relativistic version of the nuclear shell model in which the Dirac equation would be used to describe matter. More ambitiously, one may use the understanding reached with my simple model (in particular, the quantization conditions for the radius) to solve more elaborate models in which matter is also described at the field theoretic level.

Acknowledgment

I would like to acknowledge the hospitality extended to me by Walter Greiner at the University of Frankfurt.

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FIGURES

FIG. 1. Separate graphs of the three self-consistent potentials plotted for the values of the parameters given by Eq.(39) and Eq.(40). Two repulsive potentials, long range electrostatic (owing to the smallness of the fine structure constant barely different from zero on this scale) and short range mesonic, are due to the vector fields. The attractive potential is due to the scalar field.

FIG. 2. Total effective potential due to the combined effect of the three fields producing a potential well that keeps the constituents inside the bag.

FIG. 3. The density of matter ρ for a spherically symmetric bag plotted as a function of r. Note a discontinuity of the density at the bag's boundary.

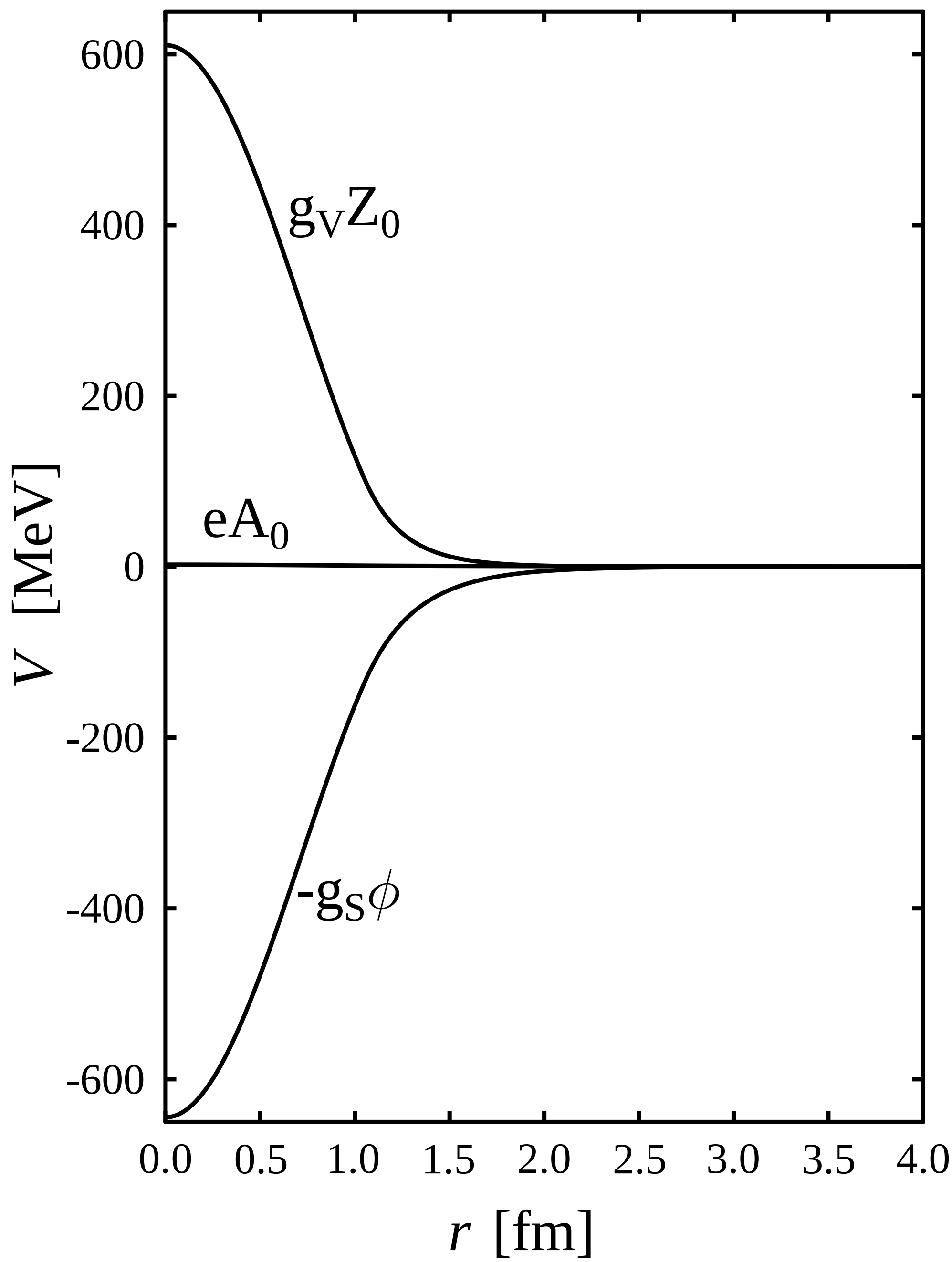

Fig. 1

Bialynicki-Birula, Simple relativistic model...

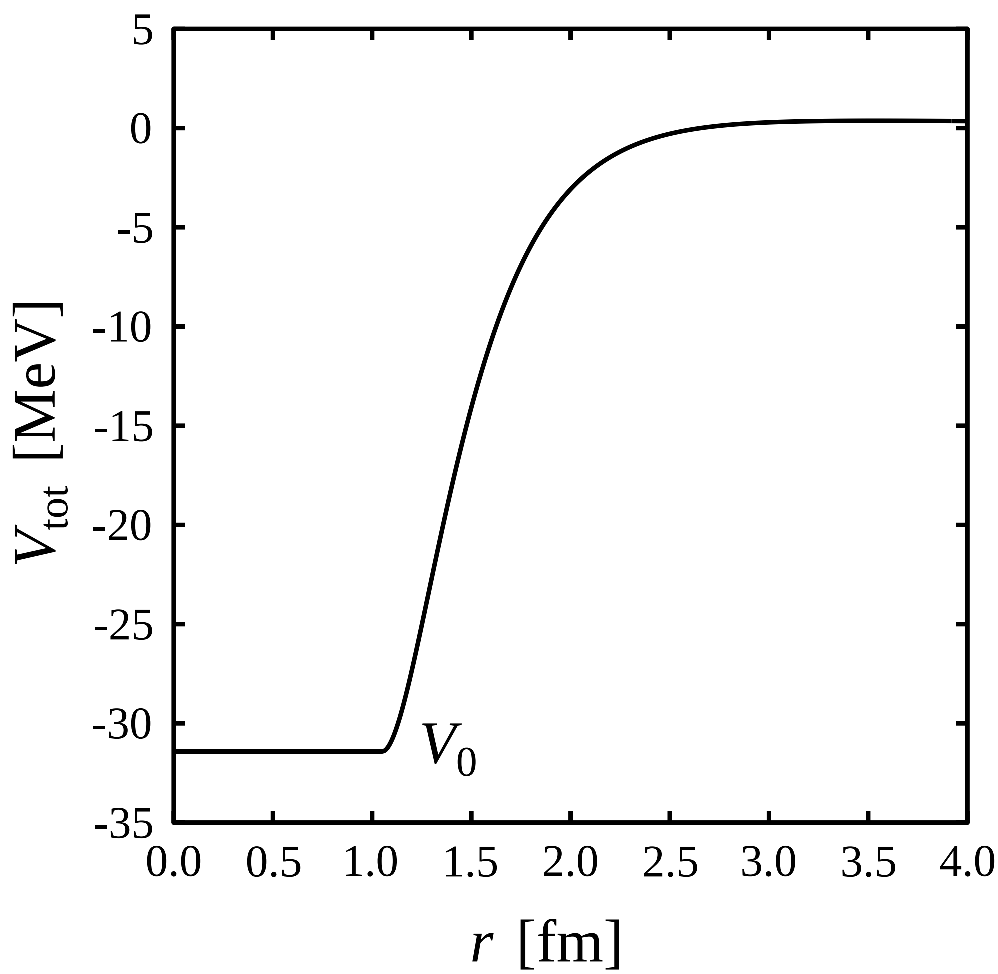

Fig. 2

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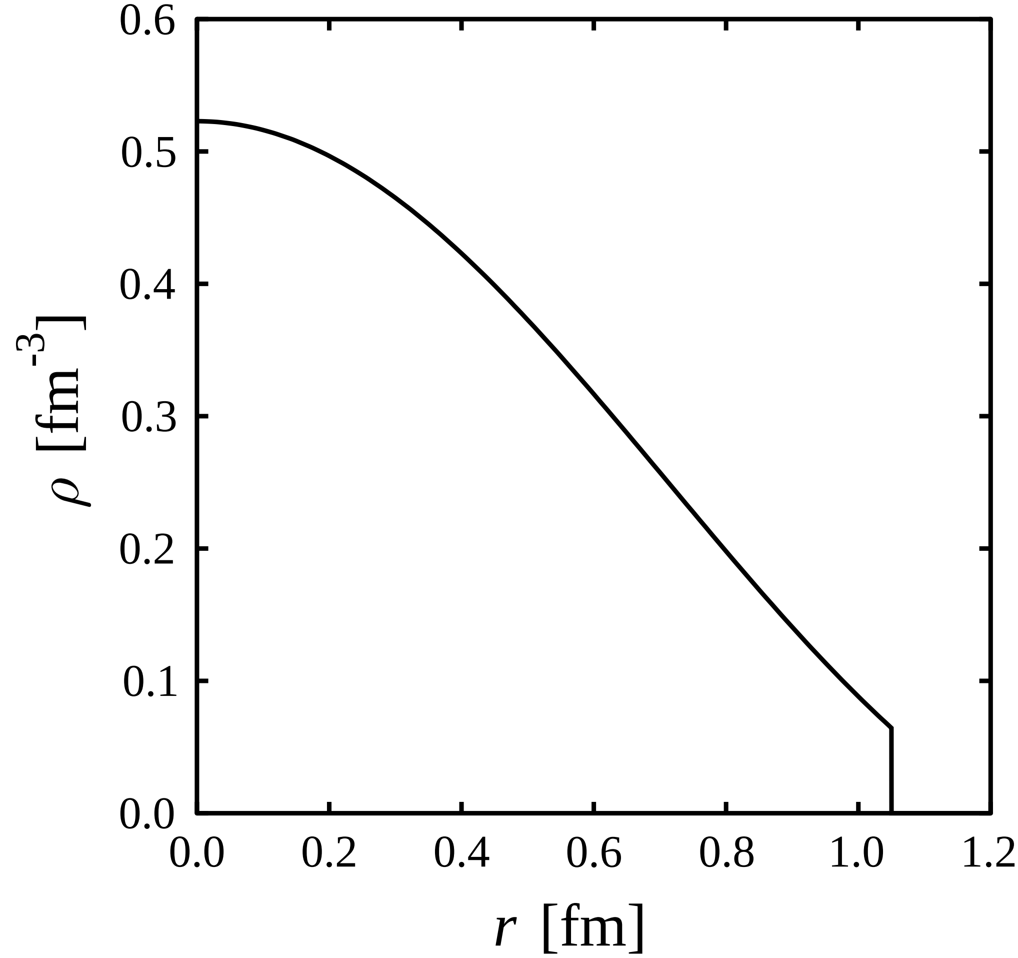

Fig. 3

Bialynicki-Birula, Simple relativistic model...